

Quadratic Formula Derivation

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1 Caveat

It is often stated that $P^2 = Q$ implies $P = \pm\sqrt{Q}$

Consider the following:

$$2^2 = Q \text{ implies } 2 = \pm\sqrt{Q}$$

Because 2^2 is always positive, we know that Q is also positive,

and therefore that $\sqrt{Q}$ is $\sqrt{+4} = 2$

But then $2 = \pm\sqrt{Q}$ is not true, since $2 \neq -2$.

2 Existence

$$Ax^2 + Bx + C = 0$$

$$\begin{aligned}(Dx + E)^2 &= D^2x^2 + 2DEx + E^2 \\ &= Ax^2 + Bx + E^2\end{aligned}$$

$$A = D^2$$

$$\text{case 1: } D = +\sqrt{A}$$

$$B = DE = +2\sqrt{A}E \rightarrow E = \frac{+B}{2\sqrt{A}}$$

$$\left(+\sqrt{A}x + \frac{B}{2\sqrt{A}}\right)^2 = Ax^2 + Bx + \frac{B^2}{4A}$$

$$Ax^2 + Bx + C = 0$$

$$Ax^2 + Bx + \frac{B^2}{4A} - \frac{B^2}{4A} + C = 0$$

$$\left(+\sqrt{A}x + \frac{B}{2\sqrt{A}}\right)^2 - \frac{B^2}{4A} + C = 0$$

$$\left(+\sqrt{A}x + \frac{B}{2\sqrt{A}}\right)^2 = \frac{B^2}{4A} - C$$

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = +\sqrt{\frac{B^2}{4A} - C}$$

or

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$$

(if $B^2 \geq 4AC$)

case 2: $D = -\sqrt{A}$

$$B = DE = -2\sqrt{A}E \rightarrow E = \frac{-B}{2\sqrt{A}}$$

$$\left(-\sqrt{A}x - \frac{B}{2\sqrt{A}}\right)^2 = Ax^2 + Bx + \frac{B^2}{4A}$$

$$Ax^2 + Bx + C = 0$$

$$Ax^2 + Bx + \frac{B^2}{4A} - \frac{B^2}{4A} + C = 0$$

$$\left(-\sqrt{A}x - \frac{B}{2\sqrt{A}}\right)^2 = \frac{B^2}{4A} - C$$

$$-\sqrt{A}x - \frac{B}{2\sqrt{A}} = +\sqrt{\frac{B^2}{4A} - C}$$

or

$$-\sqrt{A}x - \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$$

(if $B^2 \geq 4AC$)

from case 1 and case 2 we have four options:

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = +\sqrt{\frac{B^2}{4A} - C}$$

or

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$$

or

$$-\sqrt{A}x - \frac{B}{2\sqrt{A}} = +\sqrt{\frac{B^2}{4A} - C}$$

or

$$-\sqrt{A}x - \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$$

or if we multiply the bottom two equations by -1

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = +\sqrt{\frac{B^2}{4A} - C}$$

or

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$$

or

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$$

or

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = +\sqrt{\frac{B^2}{4A} - C}$$

since the first and last are the same and the second and third are the same, we are left with the following two equations

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = +\sqrt{\frac{B^2}{4A} - C}$$

or

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$$

simplifying:

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = +\sqrt{\frac{B^2}{4A} - C}$$

$$+Ax + \frac{B}{2} = +\sqrt{(\frac{B^2}{4A} - C)A}$$

$$+Ax + \frac{B}{2} = +\sqrt{\frac{B^2}{4} - AC}$$

$$+2Ax + B = +\sqrt{(\frac{B^2}{4} - AC)2^2}$$

$$+2Ax + B = +\sqrt{(\frac{B^2}{4} - AC)4}$$

$$+2Ax + B = +\sqrt{B^2 - 4AC}$$

$$+2Ax = -B + \sqrt{B^2 - 4AC}$$

$$x = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

(if $B^2 \geq 4AC$)

or

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$$

simplifying:

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$$

$$\begin{aligned}
+Ax + \frac{B}{2} &= -\sqrt{\left(\frac{B^2}{4A} - C\right)A} \\
+Ax + \frac{B}{2} &= -\sqrt{\frac{B^2}{4} - AC} \\
+2Ax + B &= -\sqrt{\left(\frac{B^2}{4} - AC\right)2^2} \\
+2Ax + B &= -\sqrt{\left(\frac{B^2}{4} - AC\right)4} \\
+2Ax + B &= -\sqrt{B^2 - 4AC} \\
+2Ax &= -B - \sqrt{B^2 - 4AC} \\
x &= \frac{-B - \sqrt{B^2 - 4AC}}{2A} \\
&(\text{if } B^2 \geq 4AC)
\end{aligned}$$

3 Uniqueness

WIP

Lemma 1:

$$x > 1 \rightarrow x^2 > x$$

$$x = 1 \rightarrow x^2 = x$$

$$0 < x < 1 \rightarrow x^2 < x$$

proof of lemma 1:

$$x = x \cdot 1$$

$$x > 1 \rightarrow \forall y > 0, y \cdot x > y \cdot 1$$

therefore,

$$x > 1 \rightarrow (x) \cdot x > (x) \cdot 1 \rightarrow x^2 > x$$

and so on for $x = 1$ and $0 < x < 1$