Quadratic Formula Derivation

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1 Caveat

It is often stated that $P^2 = Q$ implies $P = \pm\sqrt{Q}$

Consider the following:

$2^2 = Q$ implies $2 = \pm\sqrt{Q}$

Because 2^2 is always positive, we know that Q is also positive,

and therefore that $\sqrt{Q}$ is $\sqrt{+4} = 2$

But then $2 = \pm\sqrt{Q}$ is not true, since $2 \neq -2$.

2 Existence

$Ax^2 + Bx + C = 0$

$$\begin{aligned}(Dx+E)^2 &= D^2x^2 + 2DEx + E^2 \\ &= Ax^2 \;\;+\;\; Bx \;\;+\;\; E^2\end{aligned}$$

$A = D^2$

case 1: $D = +\sqrt{A}$

$B = DE = +2\sqrt{A}E \rightarrow E = \frac{+B}{2\sqrt{A}}$

$(+\sqrt{A}x + \frac{B}{2\sqrt{A}})^2 = Ax^2 + Bx + \frac{B^2}{4A}$

$Ax^2 + Bx + C = 0$

$Ax^2 + Bx + \frac{B^2}{4A} - \frac{B^2}{4A} + C = 0$

$(+\sqrt{A}x + \frac{B}{2\sqrt{A}})^2 - \frac{B^2}{4A} + C = 0$

$(+\sqrt{A}x + \frac{B}{2\sqrt{A}})^2 = \frac{B^2}{4A} - C$

$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = +\sqrt{\frac{B^2}{4A} - C}$

or

$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$

(if $B^2 \geq 4AC$)

case 2: $D = -\sqrt{A}$

$B = DE = -2\sqrt{A}E \rightarrow E = \frac{-B}{2\sqrt{A}}$

$(-\sqrt{A}x - \frac{B}{2\sqrt{A}})^2 = Ax^2 + Bx + \frac{B^2}{4A}$

$Ax^2 + Bx + C = 0$

$Ax^2 + Bx + \frac{B^2}{4A} - \frac{B^2}{4A} + C = 0$

$(-\sqrt{A}x - \frac{B}{2\sqrt{A}})^2 = \frac{B^2}{4A} - C$

$-\sqrt{A}x - \frac{B}{2\sqrt{A}} = +\sqrt{\frac{B^2}{4A} - C}$

or

$-\sqrt{A}x - \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$

(if $B^2 \geq 4AC$)

from case 1 and case 2 we have four options:

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = +\sqrt{\frac{B^2}{4A} - C}$$

or

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$$

or

$$-\sqrt{A}x - \frac{B}{2\sqrt{A}} = +\sqrt{\frac{B^2}{4A} - C}$$

or

$$-\sqrt{A}x - \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$$

or if we multiply the bottom two equations by -1

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = +\sqrt{\frac{B^2}{4A} - C}$$

or

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$$

or

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$$

or

$$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = +\sqrt{\frac{B^2}{4A} - C}$$

since the first and last are the same and the second and third are the same, we are left with the following two equations

$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = +\sqrt{\frac{B^2}{4A} - C}$

or

$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$

simplifying:

$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = +\sqrt{\frac{B^2}{4A} - C}$

$+Ax + \frac{B}{2} = +\sqrt{(\frac{B^2}{4A} - C)A}$

$+Ax + \frac{B}{2} = +\sqrt{\frac{B^2}{4} - AC}$

$+2Ax + B = +\sqrt{(\frac{B^2}{4} - AC)2^2}$

$+2Ax + B = +\sqrt{(\frac{B^2}{4} - AC)4}$

$+2Ax + B = +\sqrt{B^2 - 4AC}$

$+2Ax = -B + \sqrt{B^2 - 4AC}$

$x = \frac{-B+\sqrt{B^2-4AC}}{2A}$

(if $B^2 \geq 4AC$)

or

$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$

simplifying:

$+\sqrt{A}x + \frac{B}{2\sqrt{A}} = -\sqrt{\frac{B^2}{4A} - C}$

$+Ax + \frac{B}{2} = -\sqrt{(\frac{B^2}{4A} - C)A}$

$+Ax + \frac{B}{2} = -\sqrt{\frac{B^2}{4} - AC}$

$+2Ax + B = -\sqrt{(\frac{B^2}{4} - AC)2^2}$

$+2Ax + B = -\sqrt{(\frac{B^2}{4} - AC)4}$

$+2Ax + B = -\sqrt{B^2 - 4AC}$

$+2Ax = -B - \sqrt{B^2 - 4AC}$

$x = \frac{-B-\sqrt{B^2-4AC}}{2A}$

(if $B^2 \geq 4AC$)

3 Uniqueness

WIP

Lemma 1:

$x > 1 \; \rightarrow \; x^2 > x$

$x = 1 \; \rightarrow \; x^2 = x$

$0 < x < 1 \; \rightarrow \; x^2 < x$

proof of lemma 1:

$x = x \cdot 1$
$x > 1 \; \rightarrow \; \forall y > 0, y \cdot x > y \cdot 1$

therefore,

$x > 1 \; \rightarrow \; (x) \cdot x > (x) \cdot 1 \; \rightarrow \; x^2 > x$

and so on for $x = 1$ and $0 < x < 1$