Catalan Number Theorem

Alex Glandon

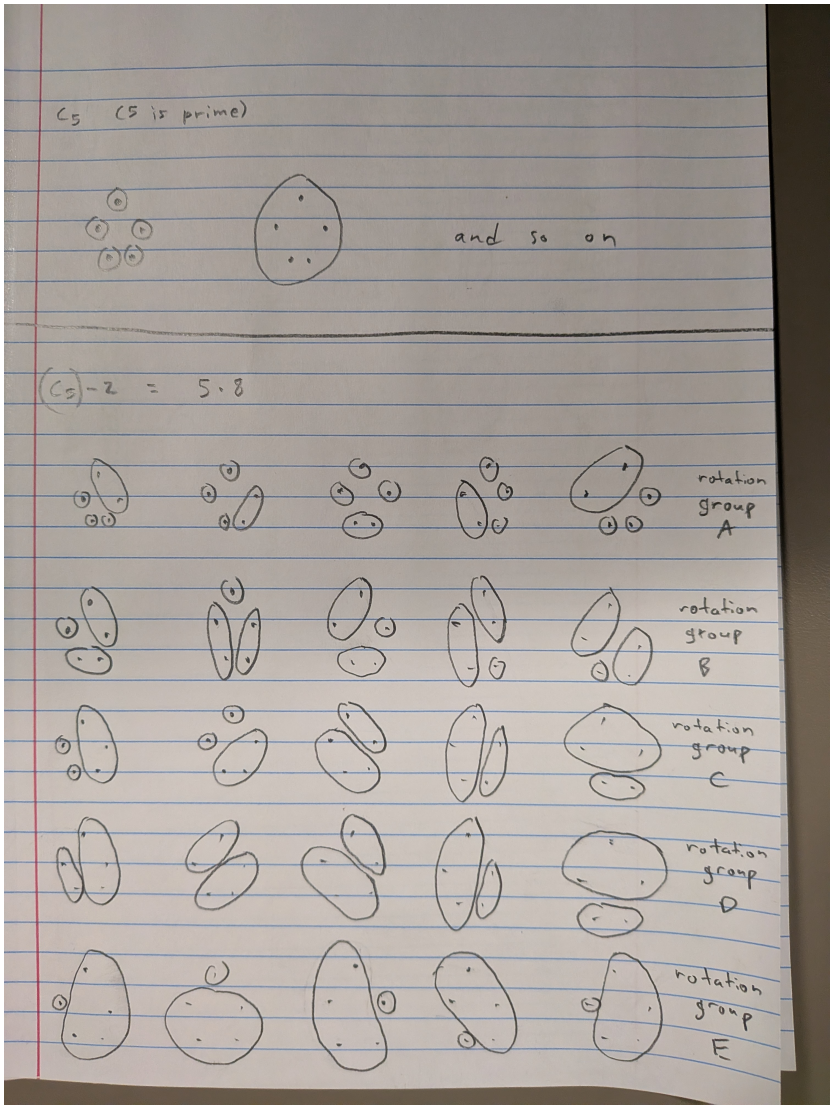
April 2026

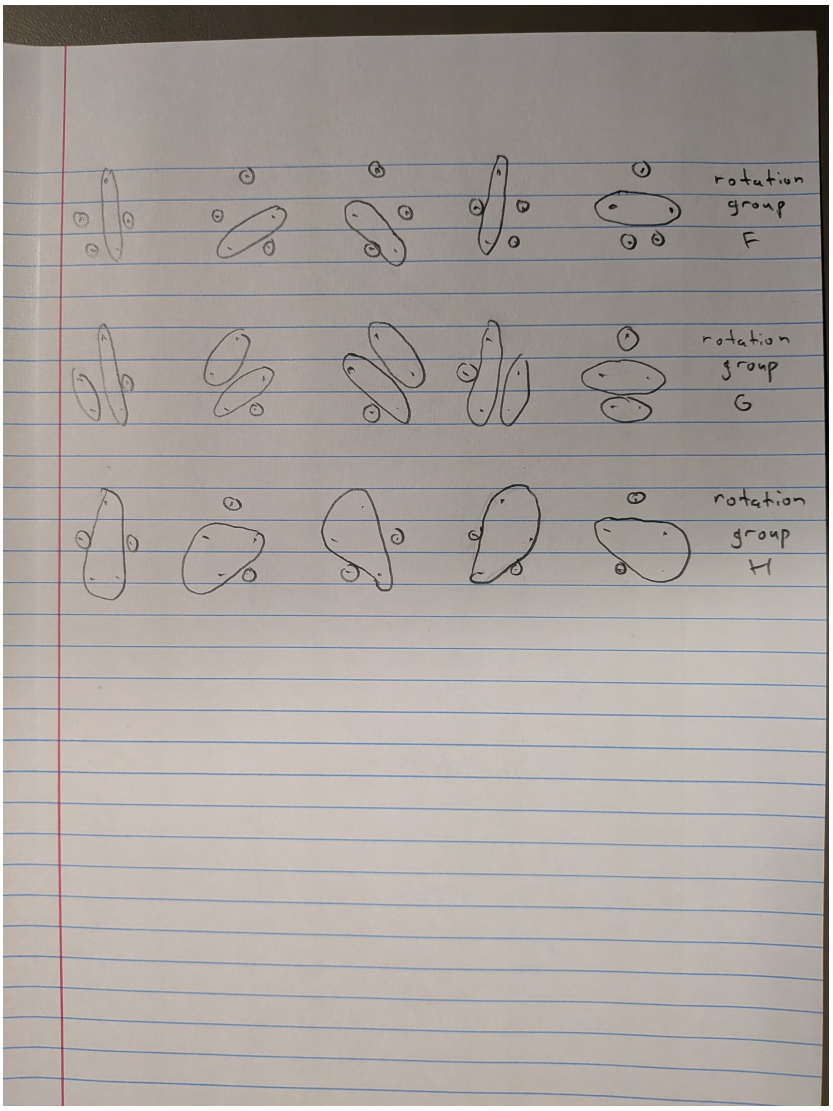
Theorem

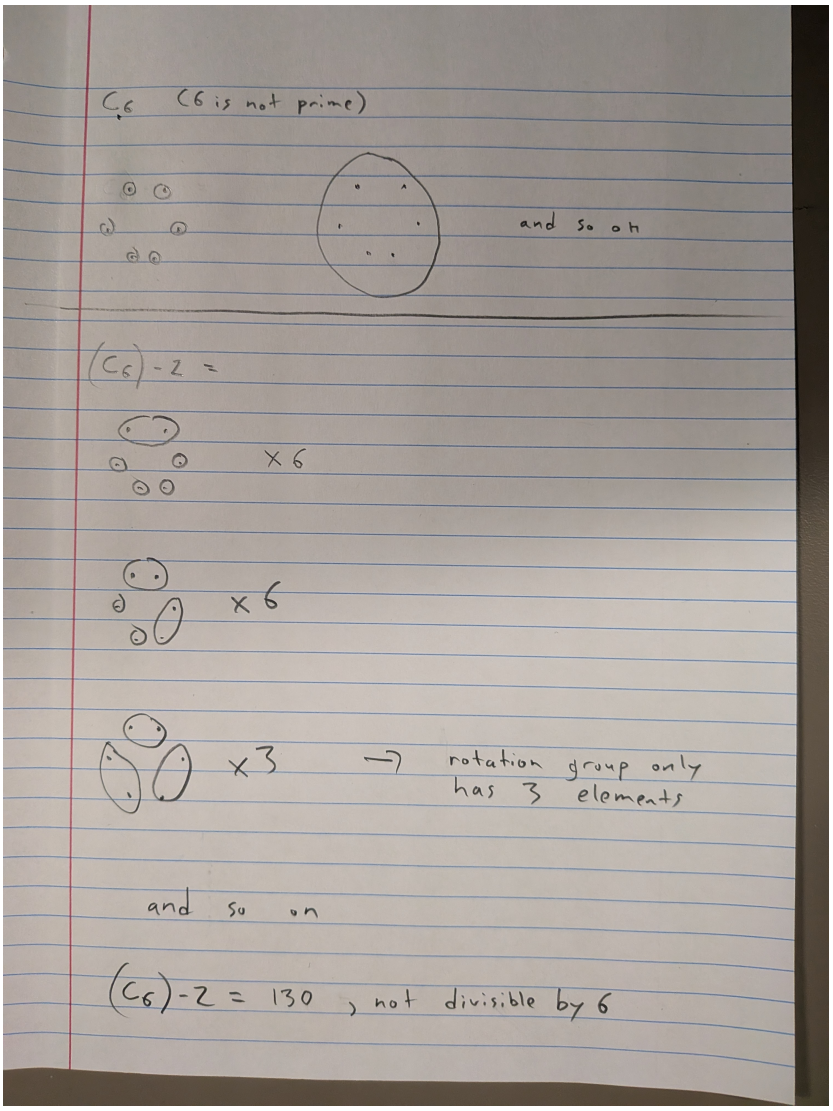
Call C_n the nth Catalan number.

If n is prime then $(C_n) - 2$ is divisible by n.

Proof 1


C_5 (5 is prime)
and so on
$(C_5)^2 = 5 \cdot 8$
rotation group A
rotation group B
rotation group C
rotation group D
rotation group E

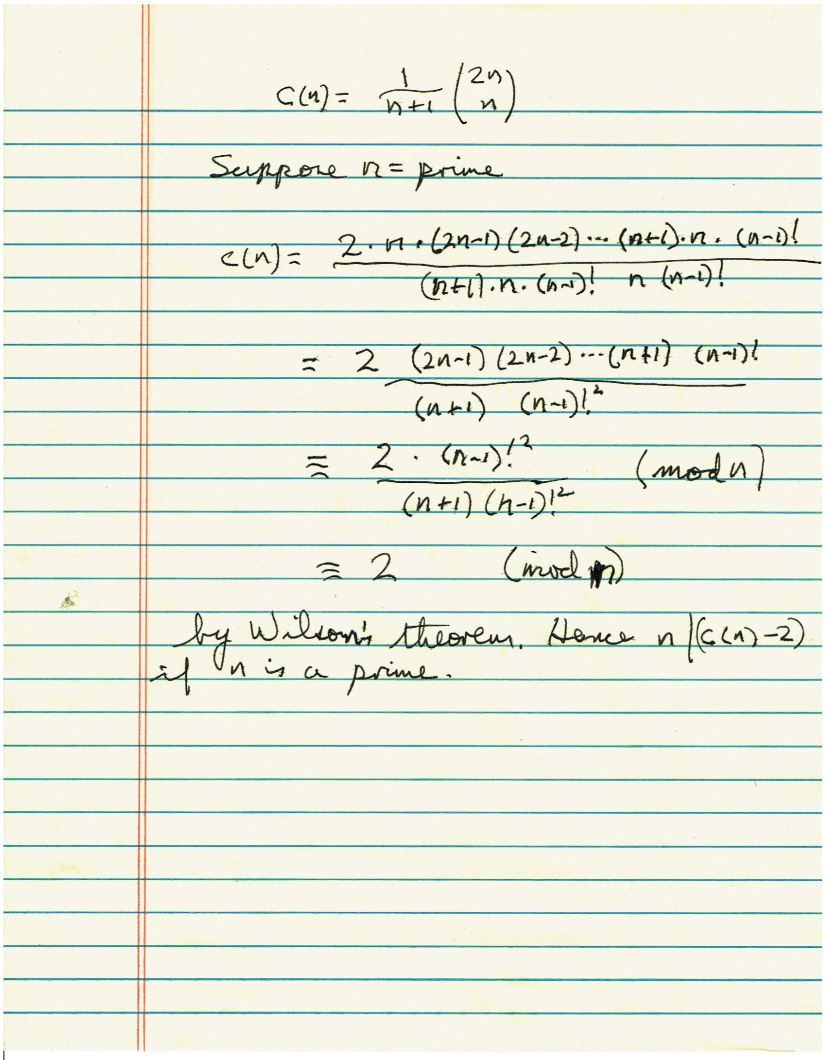

rotation group F
rotation group G
rotation group H



(for G.S. Venkatesh)

Proof 2

Another proof by Dr. George Andrews.



$$C(n) = \frac{1}{n+1}\binom{2n}{n}$$

Suppose n = prime

$$C(n) = \frac{2 \cdot n \cdot (2n-1)(2n-2)\cdots(n+1)\cdot n \cdot (n-1)!}{(n+1)\cdot n \cdot (n-1)! \quad n\,(n-1)!}$$

$$= 2\,\frac{(2n-1)(2n-2)\cdots(n+1)\,(n-1)!}{(n+1)\,(n-1)!^2}$$

$$\equiv \frac{2\cdot(n-1)!^2}{(n+1)\,(n-1)!^2} \pmod{n}$$

$$\equiv 2 \pmod{n}$$

by Wilson's theorem. Hence $n \mid (C(n)-2)$ if n is a prime.