Binomial Coefficients

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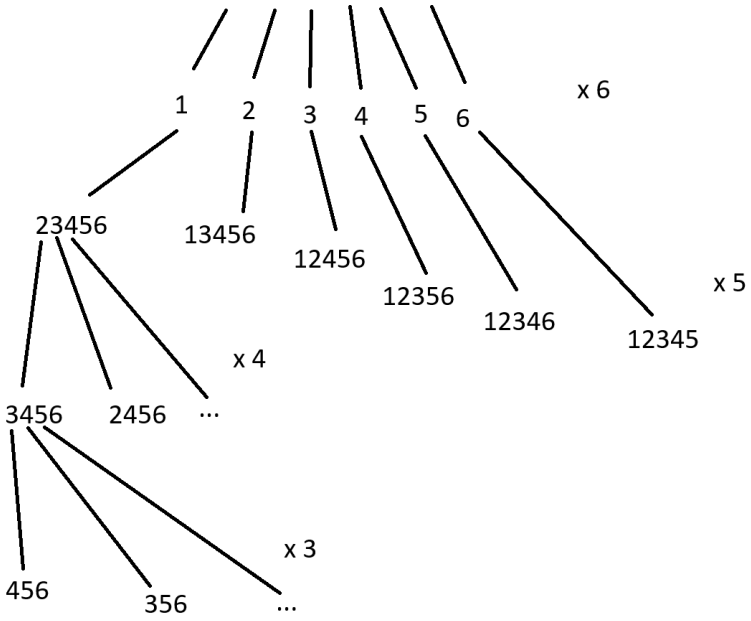
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1 Introduction

I see two ways to calculate a binomial coefficient. This is written $\binom{n}{k}$ and represents the number of ways to pick k elements from a set of n elements.

2 Rational Number Approach

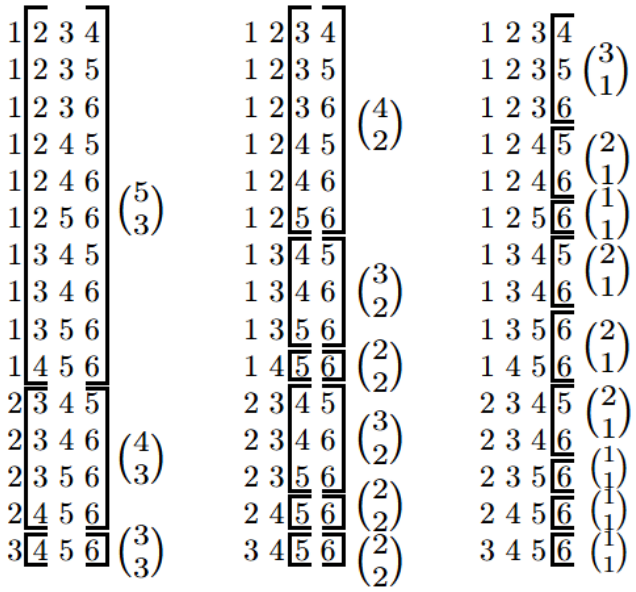
I will explain this first approach by example. For example if we are selecting 4 elements from a set of 6 elements, then we can pick elements one at a time until we have 4 elements. There would be many ways to do that. The first pick would be from any of the 6, so we would have 6 choices. The second pick would be from the remaining 5, the third pick from the remaining 4, and the fourth pick from the remaining 3. The figure below shows this:



Notice that each leaf in the tree above occurs 4! times (because the depth of the tree is 4 branches). So the graph has $6 \cdot 5 \cdot 4 \cdot 3$ leaves, but each leaf is repeated 4! times, so there are $\frac{6 \cdot 5 \cdot 4 \cdot 3}{4!}$ possible ways to select 4 elements from a graph of 6 elements.

3 Constructive Approach

I will now explain the constructive approach, again using the example of $\binom{6}{4}$. I will again use the approach of picking elements one at a time until I have four selected elements. They are picked one by one and are shown in the following figure as groups ordered from left to right. Each column shows that there are 15 ways to select 4 elements from a set of 6 elements.



This figure leads to the following recursive form.

$\binom{n}{k} =$

$\sum_{i=k-1}^{n-1} \binom{i}{k-1} =$

$\sum_{i_1=k-1}^{n-1} \sum_{i_2=k-2}^{i_1-1} \binom{i_2}{k-2} =$

$\vdots$

$\sum_{i_1=k-1}^{n-1} \sum_{i_2=k-2}^{i_1-1} \sum_{i_3=k-3}^{i_2-1} \cdots \sum_{i_{k-3}=3}^{i_{k-4}-1} \sum_{i_{k-2}=2}^{i_{k-3}-1} \sum_{i_{k-1}=1}^{i_{k-2}-1} \binom{i_{k-1}}{1} =$

$\sum_{i_1=k-1}^{n-1} \sum_{i_2=k-2}^{i_1-1} \sum_{i_3=k-3}^{i_2-1} \cdots \sum_{i_{k-3}=3}^{i_{k-4}-1} \sum_{i_{k-2}=2}^{i_{k-3}-1} \sum_{i_{k-1}=1}^{i_{k-2}-1} i_{k-1}$

For the specific case of $\binom{6}{4}$ this gives

$$((3+2+1)+(2+1)+1)+((2+1)+1)+1=15$$

compare with Pascal's triangle $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$