Quantifier Chains

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1 Existential Quantifier

$\exists x \mid 2x = 4$

The symbol $\exists$ means ”there exists”.

The above statement is true because there exists an x such that $2x = 4$, that x is 2.

Sometimes $\exists$ can be satisfied by more that one value.

$\exists x \mid x/2$ is an even number. For example 4 could be x, since $4/2$ is even or 8 could be x since $8/2$ is also even.

2 Universal Quantifier

$\forall x \in \mathbb{N},\ x + 1 \in \mathbb{N}$

The symbol $\forall$ means ”for all”.

The symbol $\in$ means lives in the space, and the symbol $\mathbb{N}$ is the space of postive integers.

So the above statement says that for all x in the positive integers, $x + 1$ is also positive, which is true.

3 Quantifier Chain Inversion

Consider a statement like $\forall$ even x, $\forall$ odd y, $\exists$ odd $z \mid x - z = y$.

This says that given any x that is even and any y that is odd, there exists a third number z that is odd and that solves $x - z = y$.

We can prove this using algebra to rewrite the equation as $x + y = z$ and noticing that if we add an even number x to and odd number y it given us an odd number z, which our statement said exists using the quantifier $\exists$.

We can rewrite a statement involving quantifiers using a chain of inversions.

Consider the following statement:

$\forall$ even x, $\forall$ odd y, $\exists$ even $z \mid x - z = y$

We know this is not true.

That means that the opposite is true, which we write using the symbol $\neg$ which says "not".

$\neg\forall$ even x, $\forall$ odd y, $\exists$ even $z \mid x - z = y$

We can rewrite this using the following two principals:

First, if $\neg\exists x \mid P(x)$ meaning there is no x that will make the statement $P(x)$ true,

Then we know $\forall x \mid \neg P(x)$, which says given any x, $P(x)$ is not true.

Second if $\neg\forall x \mid Q(x)$ meaning it is not true that for all x, we know the statement $Q(x)$ is true,

Then we know $\exists x \mid \neg Q(X)$, which says that there is at least one value of x where $Q(x)$ is false.

Now again, we want to rewrite

$\neg\forall$ even x, $\forall$ odd y, $\exists$ even z | $x - z = y$

We work through the chain, one step at a time, so first we write:

$\exists$ even x, $\neg\forall$ odd y, $\exists$ even z | $x - z = y$

Next we write:

$\exists$ even x, $\exists$ odd y, $\neg\exists$ even z | $x - z = y$

Next we write:

$\exists$ even x, $\exists$ odd y, $\forall$ even z | $\neg(x - z = y)$

So our final statement says there exists an even x and an odd y where for any even z, $x - z = y$ is not true.

For example $20 - z = 3$ is never true for any even z.

4 Waring's Problem Example

Here is another example of quantifier chain inversion. Waring's problem asks whether the following statement is true:

$\forall n$ $\exists m$ $\forall x$ $\exists$non negative integers $(a_1, a_2, \ldots, a_m)$
such that $x = a_1^n + a_2^n + \ldots + a_m^n$

or if the opposite statement is true:
$\exists n$ $\forall m$ $\exists x$ $\forall$non negative integers $(a_1, a_2, \ldots, a_m)$
such that $x \neq a_1^n + a_2^n + \ldots + a_m^n$

5 Reference

The Way of Analysis - Robert Strichartz