Euclid's Division Lemma

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1 Statement

Given $A > 0$ and a divisor $B > 0$, there exists a unique solution (C, D) with quotient $C \geq 0$ and remainder $D \geq 0$ to the equation $A/B = C + D/B$ given the condition $0 \leq D < B$, where we say C is the quotient and D is the remainder.

2 Proof 1

For example, consider the following:

$\frac{A}{B} = C + \frac{D}{B}\ as$

$\frac{1}{4} = 0 + \frac{1}{4}$

$\frac{2}{4} = 0 + \frac{2}{4}$

$\frac{3}{4} = 0 + \frac{3}{4}$

$\frac{4}{4} = 1 + \frac{0}{4}$

$\frac{5}{4} = 1 + \frac{1}{1}$

$\frac{6}{4} = 1 + \frac{1}{2}$

$\frac{7}{4} = 1 + \frac{1}{3}$

$\frac{8}{4} = 2 + \frac{2}{0}$

Then you can see for each A, B, C there is only one D such that $0 \leq D < B$.

This is equivalent to $A = B \cdot C + D$ given A, B and $0 < D \leq B$ having a unique solution C, D.

Finally, this is equivalent to $A/B = C + D/B$ given A, B and $0 \leq D < B$ having a unique solution C, D.

3 Proof 2

Consider the sequence $0B < 1B < 2B < 3B < 4B < 5B < ....$

Either $A = C \cdot B$ for some C (case 1)

or

$C \cdot B < A < (C + 1) \cdot B$ for some C. (case 2)

We see that the value of C is unique.

In the first case $A = C \cdot B + 0$ meaning $A/B = C + D/B$ where $D = 0$.

In the second case $A = C \cdot B + D$ and since A is somewhere between an integer multiple of B, then $0 \leq D < B$, meaning $A/B = C + D/B$ where $0 \leq D < B$.

4 Proof 3

Given $B > 0$, each pair of integers (C, D) such that $0 \leq D < B$ is associated with a unique integer $B \cdot C + D$

I demonstrate below by construction:

$$\begin{aligned}\mathbb{Z} = \{ \ldots, \quad & B\cdot(-1)+0, \quad B\cdot(-1)+1, \quad \ldots, \quad B\cdot(-1)+(B-1),\\ & B\cdot(\ \ 0)+0, \quad B\cdot(\ \ 0)+1, \quad \ldots, \quad B\cdot(\ \ 0)+(B-1),\\ & B\cdot(\ \ 1)+0, \quad B\cdot(\ \ 1)+1, \quad \ldots, \quad B\cdot(\ \ 1)+(B-1), \quad ... \}\end{aligned}$$

Notice, that the map between (C, D) and $B \cdot C + D$ is a bijection
and that $B \cdot C + D$ covers ever integer in $\mathbb{Z}$.

So, given $B > 0$, for any $A \in \mathbb{Z}$ there exists a unique pair (C, D), such that $0 \leq D < B$ and $A = B \cdot C + D$

5 Note

Given A, B,
$A = B \cdot C + D$ has one solution with $0 \leq D < B$.

Given A, C,
$A = B \cdot C + D$ has one solution with $0 \leq D < C$.

Given A, B, C,
$A = B \cdot C + D$ has one solution with $0 \leq D < max(B, C)$.
In other words, given an addition constraint, in order to guarantee a solution, we need to relax our other constraint on the inequality D.