Multiplication Commutativity

Alexander Glandon

July 2025

1 Introduction

This lecture describes an example of non axiomatic mathematics. We start with an inductive proof of commutativity of addition in the spirit of Peano. Then we demonstrate a non axiomatic approach.

2 Example

Let addition + be defined as an associative operator. Suppose we wanted to derive commutativity.

Define $m \times n$ as m repeated n times.

$$m \times n = \begin{array}{ll} 1 + 1 + ... + 1+ & (m\ terms) \\ 1 + 1 + ... + 1+ & (m\ terms) \\ \vdots & \\ 1 + 1 + ... + 1+ & (m\ terms) \end{array}$$

(where the above has n rows.)

We can write the product that way because of associativity

$(1 + 1) + 1 = (1 + 1) + 1 = 1 + 1 + 1$

Now we prove commutativity.

First we develop a lemma that says $m \times 1 = 1 \times m$

First we show $m \times 1 = m$

$$
\begin{array}{rcccl}
1 = 1 + 1 + ... + 1 & = & 1 + (1 + ... + 1) & = & 1 + 1 \times (m-1) \\
 & = & 2 + (1 + ... + 1) & = & 2 + 1 \times (m-2) \\
 & = & k + (1 + ... + 1) & = & k + 1 \times (m-k) \\
 & & & \vdots & \\
 & = & & & m
\end{array}
$$

$$
\begin{array}{rcl}
1 \times m & = & 1+ \\
 & & 1+ \\
 & & \vdots \\
 & & 1
\end{array}
$$

Using the same argument, this equals m.

Next we show $m \times n = n \times m$ implies $(m+1) \times n = n \times (m+1)$

$$
\begin{array}{rccc}
(m+1) \times n & = & (m+1) + (m+1) + ... + (m+1) & \\
 & = & m + m + ... + m + 1 + 1 + ... + 1 & (associativity) \\
 & = & m \times n + 1 \times n & \\
 & = & m \times n + n & \\
 & = & n \times m + n & (from\ above) \\
 & = & (n + n + ... + n) + n & (m\ terms\ and\ 1\ term) \\
 & = & n + n + ... + n + n & (m+1\ terms) \\
 & = & n \times (m+1) &
\end{array}
$$

Given that $m \times 1 = 1 \times m$ and that $m \times n = n \times m$ implies $(m+1) \times n = n \times (m+1)$, induction tells us that $m \times n = n \times m$ for any m.

Again, given $1 \times n = n \times 1$ and given $m \times n = n \times m$ implies $m \times (n+1) = (n+1) \times m$, we know that $m \times n = n \times m$ for any n.

3 Non axiomatic commutativity proof

Now again, we want to show $m \times n = n \times m$.

From the definition,

$$\begin{aligned} m \times n &= 1+1+1+1 \quad (m\ terms) \\ &+ 1+1+1+1 \quad (m\ terms) \\ &+ 1+1+1+1 \quad (m\ terms) \\ &+ 1+1+1+1 \quad (m\ terms) \\ &+ 1+1+1+1 \quad (m\ terms) \end{aligned}$$

where there are n rows.

From the definition,

$$\begin{aligned} n \times m &= 1+1+1+1+1 \quad (n\ terms) \\ &+ 1+1+1+1+1 \quad (n\ terms) \\ &+ 1+1+1+1+1 \quad (n\ terms) \\ &+ 1+1+1+1+1 \quad (n\ terms) \end{aligned}$$

where there are m rows.

Imagine you are writing the sum on a piece of paper. Notice when you rotate the paper 90 degrees, that the sum remains the same. Therefore $m \times n = n \times m$.

4 References

[1] See Shilov - Linear Algebra (material on determinants in chapter 1 for inspiration on stepping outside of an axiomatic system to do a proof).