Induction

Alex Glandon

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1 Induction

Induction is a proof method. If we can show a proposition $P(k)$ is true for $k = 1$, and we can also show that if $P(k) \to P(k+1)$, then we know $P(k)$ is true for all $k \geq 1$.

A proposition can be a statement like $2n + 1$ is odd for all n, or a proposition can be an equation like $3n + 3 = 3(n + 1)$ for all n. We will give examples of both.

2 Example 1

First we will use induction to prove a property of an expression is true.

For example, we can show $n(n^2 - 1)(3n + 2)$ is always divisible by 24 for any $n \geq 1$, meaning that if we divide $n(n^2 - 1)(3n + 2)$ by 24 then there will be no remainder.

First, we show $n(n^2 - 1)(3n + 2)$ is divisible by 24 when $n = 1$.

$1 \cdot (1^2 - 1) \cdot (3 \cdot 1 + 2) = 0$, which when divided by 24 has remainder 0.

Next we show that if $n(n^2 - 1)(3n + 2)$ is divisible by 24 when $n = k$ then $n(n^2 - 1)(3n + 2)$ is divisible by 24 when $n = k + 1$.

We note that $n(n^2 - 1)(3n + 2)_{|n=k+1} - n(n^2 - 1)(3n + 2)_{|n=k} =$

$(k + 1)((k + 1)^2 - 1)(3(k + 1) + 2) - k(k^2 - 1)(3k + 2) = 12k(k + 1)^2$.

We also note that $12k(k + 1)^2$ is always divisible by 24.

We know this because either the factor k is even or the factor $k + 1$ is even since given any two numbers is succession, either the first of the second is divisible by two.

Therefore $12k(k+1)^2$ has a factor of 12 and a factor of 2, and is therefore divisible by 24 with no remainder.

Now, if we know that $n(n^2-1)(3n+2)$ is divisible by 24 when $n = k$, then we also know that $n(n^2-1)(3n+2)$ when $n = k+1$ is equal to $k(k^2-1)(3k+2) + 12k(k+1)^2$.

Therefore $n(n^2-1)(3n+2)$ when $n = k+1$ when $n = k+1$ can be written as $24 \cdot A + 24 \cdot B$ for integers A and B since both terms are divisible by 24.

We know that $24 \cdot A + 24 \cdot B = 24(A+B)$, and therefore has a factor of 24.

Therefore, since $n(n^2-1)(3n+2)$ when $n = k+1$ equals $24(A+B)$, we know that $n(n^2-1)(3n+2)$ when $n = k+1$ is divisible by 24.

Since we have now shown that $n(n^2-1)(3n+2)$ is divisible by 24 when $n = 1$ and that if $n(n^2-1)(3n+2)$ is divisible by 24 when $n = k$ then $n(n^2-1)(3n+2)$ is divisible by 24 when $n = k+1$.

This allows us to say that $n(n^2-1)(3n+2)$ is always divisible by 24 for any $n \geq 1$. This is called proof by induction.

3 Example 2

We can also use induction to proof a formula for an expression is true.
First we find a formula for the series $\sum_{i=0}^{n-1} x^i$.

Then we again prove the formula using induction. Induction may not often be an easy way to find a formula, but sometimes it provides a good way to proof a formula for a given expression.

$$
\begin{aligned}
\sum_{i=0}^{n-1} x^i \quad = \quad x^0 + x^1 + x^2 + \cdots + x^{n-2} + x^{n-1} &= c \\
x \cdot (x^0 + x^1 + x^2 + \cdots + x^{n-2} + x^{n-1}) &= x \cdot c \\
x^1 + x^2 + x^3 + \cdots + x^{n-1} + x^n &= x \cdot c \\
x^0 + x^1 + x^2 + x^3 + \cdots + x^{n-1} + x^n &= x^0 + x \cdot c \\
x^0 + x^1 + x^2 + x^3 + \cdots + x^{n-1} &= x^0 - x^n + x \cdot c \\
c &= x^0 - x^n + x \cdot c \\
c(1-x) &= x^0 - x^n \\
c &= \tfrac{x^0 - x^n}{1-x} \qquad \text{if } x \neq 1
\end{aligned}
$$

$$c = \frac{x^n - x^0}{x-1}$$

$$c = \frac{x^n - 1}{x-1}$$

So $\sum_{i=0}^{n-1} x^i = \frac{x^n-1}{x-1}$ whenever $x \neq 1$.

We can check this formula using induction.

First we show what is called the base case, that $\sum_{i=0}^{n-1} x^i = 1$ when $n = 1$.

This is true since $\sum_{i=0}^{0} x^i = x^0 = 1$.

Next we show that if $\sum_{i=0}^{n-1} x^i = \frac{x^k-1}{x-1}$ when $n = k$, then $\sum_{i=0}^{n-1} x^i = \frac{x^{k+1}-1}{x-1}$ when $n = k+1$.

Call $\sum_{i=0}^{n-1} x^i$ as $f(n)$.

We know $f(k+1) = \sum_{i=0}^{(k+1)-1} x^i = \sum_{i=0}^{k} x^i = (\sum_{i=0}^{k-1} x^i) + x^k = f(k) + x^k$.

We also know $f(k) = \frac{x^k-1}{x-1}$

Therefore $f(k+1) = f(k)+x^k = \frac{x^k-1}{x-1}+x^k = \frac{x^k-1}{x-1}+\frac{(x-1)x^k}{x-1} = \frac{x^k-1}{x-1}+\frac{x^{k+1}-x^k}{x-1} = \frac{x^k-1+x^{k+1}-x^k}{x-1} = \frac{x^{k+1}-1}{x-1}$, which proves that $f(n) = \frac{x^{k+1}-1}{x-1}$ when $n = k+1$ given that $f(n) = \frac{x^k-1}{x-1}$ when $n = k$.

This completes our proof of the formula $\sum_{i=0}^{n-1} x^i = \frac{x^n-1}{x-1}$ using induction.

We will use this formula later in lecture 6 to find an alternate proof of $n < k^n$ when $k \geq 2$.

4 Example 3

Next we will show the formulas for $\sum_{i=1}^{n} i^m$ and then check the formulas using induction.

First we will use a method from Apostol's Calculus textbook.

First we use $(k-1)^4 = k^4 - 4k^3 + 6k^2 - 4k + 1$

Then $k^4 - (k-1)^4 = k^4 - (k^4 - 4k^3 + 6k^2 - 4k + 1) = 4k^3 - 6k^2 + 4k - 1$

Now, consider the series of equations using the above form:

$1^4 - 0^4 = 4 \cdot 1^3 - 6 \cdot 1^2 + 4 \cdot 1 - 1$
$2^4 - 1^4 = 4 \cdot 2^3 - 6 \cdot 2^2 + 4 \cdot 2 - 1$
$3^4 - 2^4 = 4 \cdot 3^3 - 6 \cdot 3^2 + 4 \cdot 3 - 1$
$\vdots$
$k^4 - (k-1)^4 = 4k^3 - 6k^2 + 4k - 1$
$\vdots$
$n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1$

If we add all of the terms on the left we get

$(1^4 - 0^4) + (2^4 - 1^4) + (3^4 - 2^4) + \cdots + (n^4 - (n-1)^4) =$

$n^4 + (-(n-1)^4 + (n-1)^4) + \cdots + (-3^4 + 3^4) + (-2^4 + 2^4) + (-1^4 + 1^4) + 0^4 =$

$n^4 - 0^4 =$

n^4

If we add all of the terms on the right we get

$4 \cdot \sum_{i=1}^n i^3 - 6 \cdot \sum_{i=1}^n i^2 + 4 \cdot \sum_{i=1}^n i - \sum_{i=1}^n 1 =$

$4 \cdot \sum_{i=1}^n i^3 - 6 \cdot \sum_{i=1}^n i^2 + 4 \cdot \sum_{i=1}^n i - n =$

Therefore

$n^4 = 4 \cdot \sum_{i=1}^n i^3 - 6 \cdot \sum_{i=1}^n i^2 + 4 \cdot \sum_{i=1}^n i - n$

This shows that if we had formulas for $\sum_{i=1}^n i^2$ and $\sum_{i=1}^n i$ that we could find a formula for $\sum_{i=1}^n i^3$.

Using a similar procedure, we can find a formula for $\sum_{i=1}^n i^2$ in terms of $\sum_{i=1}^n i$.

And finally, we can use the same procedure on the base case:

$n^2 - (n-1)^2 = n^2 - (n^2 - 2n + 1) = 2n - 1$

Therefore

$1^2 - 0^2 = 2 \cdot 1 - 1$
$2^2 - 1^2 = 2 \cdot 2 - 1$
$3^2 - 2^2 = 2 \cdot 3 - 1$
$\vdots$
$n^2 - (n-1)^2 = 2n - 1$

$n^2 - 0^2 = 2 \cdot \sum_{i=1}^{n} i - n$

$n^2 = 2 \cdot \sum_{i=1}^{n} i - n$

$n^2 + n = 2 \cdot \sum_{i=1}^{n} i$

$n(n+1) = 2 \cdot \sum_{i=1}^{n} i$

$\frac{n(n+1)}{2} = \sum_{i=1}^{n} i$

This procedure produces a polynomial on the base case, and produces a formula for $\sum_{i=1}^{n} i^m$ in terms of $\sum_{i=1}^{n} i^{m-1}, \sum_{i=1}^{n} i^{m-2}, \cdots, \sum_{i=1}^{n} i$ and n^{m+1}. Because the sum of polynomials of n with highest order $m+1$ is a polynomial of n with highest order $m+1$ (we know the $m+1$ term is not canceled out because the only term of order $m+1$ is n^{m+1}), we can use induction to say that the formula for $\sum_{i=1}^{n} i^m$ is a polynomial of order $m+1$.

Because we know that $\sum_{i=1}^{n} i^m$ is a polynomial of order $m+1$, we can solve for the coefficients of $\sum_{i=1}^{n} i^m$ without needing to find $\sum_{i=1}^{n} i^{m-1}, \sum_{i=1}^{n} i^{m-2}, \cdots, \sum_{i=1}^{n} i$ first.

First lets solve for $\sum_{i=1}^{n} i^3$ using the long approach to show the benefit of a more direct solution following.

We already showed $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Now we use $n^3 - (n-1)^3 = n^3 - (n^3 - 3n^2 + 3n - 1) = 3n^2 - 3n + 1$

Therefore

$1^3 - 0^3 = 3 \cdot 1^2 - 3 \cdot 1 + 1$
$2^3 - 1^3 = 3 \cdot 2^2 - 3 \cdot 2 + 1$
$3^3 - 2^3 = 3 \cdot 3^2 - 3 \cdot 3 + 1$
$\vdots$
$n^3 - (n-1)^3 = 3n^2 - 3n + 1$

Adding the left and right we get

$n^3 - 0^3 = 3 \cdot \sum_{i=1}^{n} i^2 - 3 \cdot \sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$

$n^3 = 3 \cdot \sum_{i=1}^{n} i^2 - 3 \cdot \sum_{i=1}^{n} i + n$

$n^3 = 3 \cdot \sum_{i=1}^{n} i^2 - 3\frac{n(n+1)}{2} + n$

$2n^3 = 6 \cdot \sum_{i=1}^{n} i^2 - 3n(n+1) + 2n$

$2n^3 + 3n(n+1) - 2n = 6 \cdot \sum_{i=1}^{n} i^2$

$2n^3 + 3n^2 + 3n - 2n = 6 \cdot \sum_{i=1}^{n} i^2$

$2n^3 + 3n^2 + n = 6 \cdot \sum_{i=1}^{n} i^2$

$n(n+1)(2n+1) = 6 \cdot \sum_{i=1}^{n} i^2$

$\frac{n(n+1)(2n+1)}{6} = \sum_{i=1}^{n} i^2$

Finally now that we have $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ we can solve for $\sum_{i=1}^{n} i^3$ from earlier.

$n^4 = 4 \cdot \sum_{i=1}^{n} i^3 - 6 \cdot \sum_{i=1}^{n} i^2 + 4 \cdot \sum_{i=1}^{n} i - n$

$n^4 = 4 \cdot \sum_{i=1}^{n} i^3 - 6 \cdot \frac{n(n+1)(2n+1)}{6} + 4 \cdot \frac{n(n+1)}{2} - n$

$n^4 = 4 \cdot \sum_{i=1}^{n} i^3 - n(n+1)(2n+1) + 2n(n+1) - n$

$n^4 = 4 \cdot \sum_{i=1}^{n} i^3 - 2n^3 - 3n^2 - n + 2n^2 + 2n - n$

$n^4 = 4 \cdot \sum_{i=1}^{n} i^3 - 2n^3 - n^2$

$n^4 + 2n^3 + n^2 = 4 \cdot \sum_{i=1}^{n} i^3$

$n^2(n+1)^2 = 4 \cdot \sum_{i=1}^{n} i^3$

$\frac{n^2(n+1)^2}{4} = \sum_{i=1}^{n} i^3$

We can also notice

$(\frac{n(n+1)}{2})^2 = \sum_{i=1}^{n} i^3$

And therefore $(\sum_{i=1}^{n} i)^2 = \sum_{i=1}^{n} i^3$

Now that we solved for $\sum_{i=1}^{n} i^3$ the longer way, we can use the fact that $\sum_{i=1}^{n} i^3$ is a polynomial of order 4 to find a shorter solution. This would be especially helpful is we needed high order m in $\sum_{i=1}^{n} i^m$

We start with

$\sum_{i=1}^{1} i^3 = 1^3 = 1$

$\sum_{i=1}^{1} i^3 = 1^3 + 2^3 = 9$

$\sum_{i=1}^{1} i^3 = 1^3 + 2^3 + 3^3 = 36$

$\sum_{i=1}^{1} i^3 = 1^3 + 2^3 + 3^3 + 4^3 = 100$
$\sum_{i=1}^{1} i^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 225$

We also know $\sum_{i=1}^{n} i^3 = a_1 n^4 + a_2 n^3 + a_3 n^2 + a_4 n + a_5$ since we saw earlier that it is a polynomial of order 4

so that

$\sum_{i=1}^{1} i^3 = a_1 \cdot 1^4 + a_2 \cdot 1^3 + a_3 \cdot 1^2 + a_4 \cdot 1 + a_5$

$\sum_{i=1}^{2} i^3 = a_1 \cdot 2^4 + a_2 \cdot 2^3 + a_3 \cdot 2^2 + a_4 \cdot 2 + a_5$

$\sum_{i=1}^{3} i^3 = a_1 \cdot 3^4 + a_2 \cdot 3^3 + a_3 \cdot 3^2 + a_4 \cdot 3 + a_5$

$\sum_{i=1}^{4} i^3 = a_1 \cdot 4^4 + a_2 \cdot 4^3 + a_3 \cdot 4^2 + a_4 \cdot 4 + a_5$

$\sum_{i=1}^{5} i^3 = a_1 \cdot 4^4 + a_2 \cdot 4^3 + a_3 \cdot 4^2 + a_4 \cdot 4 + a_5$

This gives us a linear system of equations.

$$\begin{bmatrix} 1 & 1 & 1^2 & 1^3 & 1^4 \\ 1 & 2 & 2^2 & 2^3 & 2^4 \\ 1 & 3 & 3^2 & 3^3 & 3^4 \\ 1 & 4 & 4^2 & 4^3 & 4^4 \\ 1 & 5 & 5^2 & 5^3 & 5^4 \end{bmatrix} \begin{bmatrix} a_5 \\ a_4 \\ a_3 \\ a_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{1} i^3 \\ \sum_{i=1}^{2} i^3 \\ \sum_{i=1}^{3} i^3 \\ \sum_{i=1}^{4} i^3 \\ \sum_{i=1}^{5} i^3 \end{bmatrix}$$

We can evaluate the right sums

$$\begin{bmatrix} 1 & 1 & 1^2 & 1^3 & 1^4 \\ 1 & 2 & 2^2 & 2^3 & 2^4 \\ 1 & 3 & 3^2 & 3^3 & 3^4 \\ 1 & 4 & 4^2 & 4^3 & 4^4 \\ 1 & 5 & 5^2 & 5^3 & 5^4 \end{bmatrix} \begin{bmatrix} a_5 \\ a_4 \\ a_3 \\ a_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1+8 \\ 1+8+27 \\ 1+8+27+64 \\ 1+8+27+64+125 \end{bmatrix}$$

and simplify to

$$\begin{bmatrix} 1 & 1 & 1^2 & 1^3 & 1^4 \\ 1 & 2 & 2^2 & 2^3 & 2^4 \\ 1 & 3 & 3^2 & 3^3 & 3^4 \\ 1 & 4 & 4^2 & 4^3 & 4^4 \\ 1 & 5 & 5^2 & 5^3 & 5^4 \end{bmatrix} \begin{bmatrix} a_5 \\ a_4 \\ a_3 \\ a_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 36 \\ 100 \\ 225 \end{bmatrix}$$

When we arrange it like this, the matrix above on the left is an example of a Vandermonde matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 & 16 \\ 1 & 3 & 9 & 27 & 81 \\ 1 & 4 & 16 & 64 & 256 \\ 1 & 5 & 25 & 125 & 625 \end{bmatrix} \begin{bmatrix} a_5 \\ a_4 \\ a_3 \\ a_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 36 \\ 100 \\ 225 \end{bmatrix}$$

We want to evaluate this analytically, so we get rational numbers for the coefficients of the polynomial.

Using Gaussian elimination

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 & 15 \\ 0 & 2 & 8 & 26 & 80 \\ 0 & 3 & 15 & 63 & 255 \\ 0 & 4 & 24 & 124 & 624 \end{bmatrix} \begin{bmatrix} a_5 \\ a_4 \\ a_3 \\ a_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 35 \\ 99 \\ 224 \end{bmatrix} \begin{matrix} \\ \text{(row 2 - row 1)} \\ \text{(row 3 - row 1)} \\ \text{(row 4 - row 1)} \\ \text{(row 5 - row 1)} \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 & 15 \\ 0 & 0 & 2 & 12 & 50 \\ 0 & 0 & 6 & 42 & 210 \\ 0 & 0 & 12 & 96 & 564 \end{bmatrix} \begin{bmatrix} a_5 \\ a_4 \\ a_3 \\ a_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 19 \\ 75 \\ 192 \end{bmatrix} \begin{matrix} \\ \\ \text{(row 3 - 2} \times \text{row 2)} \\ \text{(row 4 - 3} \times \text{row 2)} \\ \text{(row 5 - 4} \times \text{row 2)} \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 & 15 \\ 0 & 0 & 2 & 12 & 50 \\ 0 & 0 & 0 & 6 & 60 \\ 0 & 0 & 0 & 24 & 264 \end{bmatrix} \begin{bmatrix} a_5 \\ a_4 \\ a_3 \\ a_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 19 \\ 18 \\ 78 \end{bmatrix} \begin{matrix} \\ \\ \\ \text{(row 4 - 3} \times \text{row 3)} \\ \text{(row 5 - 6} \times \text{row 3)} \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 & 15 \\ 0 & 0 & 2 & 12 & 50 \\ 0 & 0 & 0 & 6 & 60 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix} \begin{bmatrix} a_5 \\ a_4 \\ a_3 \\ a_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ 19 \\ 18 \\ 6 \end{bmatrix} \begin{matrix} \\ \\ \\ \\ \text{(row 5 - 4} \times \text{row 4)} \end{matrix}$$

Then $24 \cdot a_1 = 6$,
so $a_1 = 1/4$

$6 \cdot a_2 + 60 \cdot a_1 = 18$,
$6 \cdot a_2 + 15 = 18$,
$6 \cdot a_2 = 3$,
$a_2 = 1/2$

$2 \cdot a_3 + 12 \cdot a_2 + 50 \cdot a_1 = 19$,
$2 \cdot a_3 + 6 + 25/2 = 19$,
$4 \cdot a_3 + 12 + 25 = 38$,
$4 \cdot a_3 = 1$,
$a_3 = 1/4$,

$a_4 + 3 \cdot a_3 + 7 \cdot a_2 + 15 \cdot a_1 = 8$,
$a_4 + 3/4 + 7/2 + 15/4 = 8$,
$4 \cdot a_4 + 3 + 14 + 15 = 32$,
$4 \cdot a_4 = 0$,
$a_4 = 0$

$a_5 + a_4 + a_3 + a_2 + a_1 = 1$,
$a_5 + 0 + 1/4 + 1/2 + 1/4 = 1$,
$a_5 = 0$

So the polynomial we were looking for was

$\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 =$

$\frac{1}{4}(n^4 + 2n^3 + n^2) =$

$\frac{n^2(n+1)^2}{4}$

which is what we found earlier.

Finally, we can prove $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$ using induction.

The base case is $\sum_{i=1}^{1} i^3 = \frac{1^2(1+1)^2}{4}$, $1 = 1$

Then we need to show $\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$ implies $\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$

If $\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$,

then $\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 =$

$\frac{k^4+2k^3+k^2}{4} + k^3 + 3k^2 + 3k + 1 =$

$\frac{k^4+2k^3+k^2}{4} + \frac{4k^3+12k^2+12k+4}{4} =$

$\frac{k^4+2k^3+k^2+4k^3+12k^2+12k+4}{4} =$

$\frac{k^4+6k^3+13k^2+12k+4}{4} =$

$\frac{(k^2+2k+1)(k^2+4k+4)}{4} =$

$\frac{(k+1)^2(k+2)^2}{4} =$

$\frac{(k+1)^2((k+1)+1)^2}{4}$

which is what we set out to prove.